ON THE REPLICATIONS OF CERTAIN MULTIPLICATIVE DESIGNS

BY

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ABSTRACT

Bounds on the number of row sums in an $n \times n$, non-singular (0,1)-matrix A sarisfying $A^tA = \text{diag}\,(k_1 - \lambda_1, \cdots, k_n - \lambda_n)$, $k_j > \lambda_j > 0$, $\lambda_1 = \cdots = \lambda_e$, $\lambda_{e+1} = \cdots = \lambda_n$ are obtained which extend previous results for such matrices.

1. Introduction

A multiplicative design [3] is a combinatorial configuration consisting of n subsets of an n-set whose $(0,1)$-incidence matrix, A, satisfies

$$A^tA = D + \left(\sqrt{\lambda_i\lambda_j}\right) \tag{1.1}$$

where $D = \text{diag}(k_1 - \lambda_1, k_2 - \lambda_2, \cdots, k_n - \lambda_n)$ and $k_j > \lambda_j > 0$. In [3], Ryser obtains parameter relations and some structure results for such designs which are generalizations of the (v, k, λ)-configurations [4] and λ-design [1, 2, 5] results. Special cases considered are "uniform" designs where D is a scalar matrix and the case where the parameters λ_i satisfy $\lambda_2 = \lambda_3 = \cdots = \lambda_n$. In this latter case, it is shown that there are at most four replications (row sums) in the design.

In this paper we generalize this last theorem and incorporate, as well, the two-replication result for λ-designs in the following theorem.

THEOREM 1.1. *Let A be the incidence matrix of a multiplicative design satisfying* (1.1) *with $\lambda_1 = \lambda_2 = \cdots = \lambda_e$, $\lambda_{e+1} = \lambda_{e+2} = \cdots = \lambda_n$. Then A has at most 2^{e+1} row sums. Moreover if the design is partially uniform, in the sense that $k_1 = k_2 = \cdots = k_e$ then A can have at most $2(e+1)$ row sums.*

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2. Proof of Theorem 1.1

We use the following notation:

$$(2.1)\quad \begin{gathered} \pi = \prod_{j=1}^{n} (k_j - \lambda_j),\ \pi_i = \pi/(k_i - \lambda_i),\ \boldsymbol{\pi} = (\pi_1\sqrt{\lambda_1}, \pi_2\sqrt{\lambda_2}, \cdots, \pi_n\sqrt{\lambda_n})^t, \\ \Delta = \pi + \sum_{j=1}^{n} \lambda_j \pi_j = \det\left(D + \left(\sqrt{\lambda_i\lambda_j}\right)\right), \end{gathered}$$

and note that a straightforward calculation verifies that

$$(2.2)\qquad \left(D + \left(\sqrt{\lambda_i\lambda_j}\right)\right)^{-1} = D^{-1} - \frac{1}{\Delta\pi}\boldsymbol{\pi}\boldsymbol{\pi}^t$$

and

$$(2.3)\qquad A\,{}^t\!A\boldsymbol{\pi} = \Delta\,(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots, \sqrt{\lambda_n})^t.$$

From (1.1) and (2.2), one obtains directly the basic relation exhibited in Ryser [3]:

$$(2.4)\qquad AD^{-1}A^t = I + \frac{\Delta}{\pi}(x_i x_j)$$

where

$$(2.5)\qquad x_j = \frac{\pi}{\Delta} \sum_{i=1}^{n} \frac{a_{ji}\sqrt{\lambda_i}}{k_i - \lambda_i}.$$

We next observe that multiplication of (1.1) by a column vector of ones yields

$$(2.6)\qquad A^t\boldsymbol{r} = \left(\sum_{i=1}^{n} \sqrt{\lambda_i}\right)\left(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots, \sqrt{\lambda_n}\right)^t - (\lambda_1, \lambda_2, \cdots, \lambda_n)^t$$

where $\boldsymbol{r} = (r_1 - 1, r_2 - 1, , r_n - 1)^t$, r_j being the sum of row j of A. In view of (2.3) we seek to construct another vector, not $\boldsymbol{r}$, which also satisfies (2.6), supposing from here on that $\lambda_1 = \lambda_2 = \cdots = \lambda_e$ and $\lambda_{e+1} = \cdots = \lambda_n$.

Define

$$(2.7)\qquad c_i = \frac{\sqrt{\lambda_1\lambda_n} - \lambda_1}{k_i - \lambda_1} \qquad (i = 1, \cdots, e)$$

and

$$(2.8)\qquad s = \left(\sqrt{\lambda_1} - \sqrt{\lambda_n}\right)\left(e + \lambda_1 \sum_{i=1}^{e} \frac{1}{k_i - \lambda_1}\right) + (n-1)\sqrt{\lambda_n}.$$

One may then verify that the numbers c_i and s satisfy

$$(2.9)\quad \begin{bmatrix} k_1 & \lambda_1 & \cdots\ \lambda_1 & \sqrt{\lambda_1} \\ \lambda_1 & k_2 & \lambda_1 \cdots \lambda_1 & \sqrt{\lambda_1} \\ \vdots & \ddots & & \vdots \\ \lambda_1 & \cdots & \lambda_1\ \ k_e & \sqrt{\lambda_1} \\ \sqrt{\lambda_1\lambda_n} & \sqrt{\lambda_1\lambda_n} & \cdots \sqrt{\lambda_1\lambda_n} & \sqrt{\lambda_n} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_e \\ s \end{bmatrix} = \sum_{i=1}^{n} \sqrt{\lambda_i} \begin{bmatrix} \sqrt{\lambda_1} \\ \sqrt{\lambda_1} \\ \vdots \\ \sqrt{\lambda_1} \\ \sqrt{\lambda_n} \end{bmatrix} - \begin{bmatrix} \lambda_1 \\ \lambda_1 \\ \vdots \\ \lambda_1 \\ \lambda_n \end{bmatrix}$$

We now set

$$Y = \frac{s}{\Delta} A\pi + \sum_{i=1}^{e} c_i A_i$$

where A_i is the ith column vector of A and observe that (2.3) and (2.9) imply

$$\begin{aligned} A^t Y &= s\left(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n}\right)^t + \sum_{i=1}^{e} c_i A^t A_i \\ &= \left(\sum_{i=1}^{n} \sqrt{\lambda_i}\right)\left(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n}\right)^t - (\lambda_1, \cdots, \lambda_n)^t. \end{aligned}$$

This means, since A is non-singular, that $Y = r$. Using $p_j = \sum_{i=1}^{e} \frac{a_{ji}}{k_i - \lambda_1}$ and x_j as in (2.5), $Y = r$ is the assertion

(2.10) $$r_j - 1 = sx_j + \left(\sqrt{\lambda_1 \lambda_n} - \lambda_1\right) p_j, \qquad j = 1, \cdots, n.$$

We next obtain a relation between x_j and p_j as follows. Read the j, j positions in the Eq. (2.4) in view of $a_{jl}^2 = a_{jl}$:

(2.11) $$\sum_{l=1}^{n} \frac{a_{jl}}{k_l - \lambda_l} = 1 + \frac{\Delta}{\pi} x_j^2.$$

The definition of x_j implies

(2.12) $$x_j = \frac{\pi}{\Delta} \sqrt{\lambda_1}\, p_j + \frac{\pi}{\Delta} \sqrt{\lambda_n} \sum_{i=e+1}^{n} \frac{a_{ji}}{k_i - \lambda_n}$$

and (2.11) can be written

(2.13) $$p + \sum_{i=e+1}^{n} \frac{a_{ji}}{k_i - \lambda_n} = 1 + \frac{\Delta}{\pi} x_j^2 .$$

Elimination of the indicated summation in (2.12) and (2.13) yields

(2.14) $$x_j^2 - \frac{1}{\sqrt{\lambda_n}} x_j + \frac{\pi}{\Delta}\left[1 + \left(\frac{\sqrt{\lambda_1} - \sqrt{\lambda_n}}{\sqrt{\lambda_n}}\right) p_j\right] = 0.$$

Relations (2.14) and (2.10) give the first part of the theorem, for if we consider those rows of A for which $p_j = \bar{p}$, they say there are at most two possible row sums among these rows. For a given design there are evidently at most 2^e values possible for p_j, so that A has at most $2 \cdot 2^e = 2^{e+1}$ replications. In case $k_1 = k_2$

$= \cdots = k_e$, we have that $p_j = r'_{j'}(k_1 - \lambda_1)$ where r'_j is the partial sum of row j over the first e columns. Hence there are at most $(e + 1)$ p_j values.

We conclude with an example of a multiplicative design with $\lambda_1 = \cdots = \lambda_e$, $\lambda_{e+1} = \cdots = \lambda_n$ having several replications. Take the incidence matrix A_1 of a λ-design of order n. With $t = \lambda + 1$, let A_2 be the incidence matrix of a symmetric block design with parameters $(\mu t^2 - t + 1, \mu t, \mu)$ and form the matrix

$$A = \begin{bmatrix} 0 & A_1 \\ A_2 & J \end{bmatrix}$$

where 0 is the zero matrix of size $n \times (\mu t^2 - t + 1)$ and J is the matrix of ones of size $(\mu t^2 - t + 1) \times n$. Then A is the incidence matrix of a multiplicative design with $\lambda_1 = \cdots = \lambda_{\mu t^2 - t + 1} = \mu$, $\lambda_{t^2 - t + 2} = \cdots = \lambda_{n + \mu t^2 - t + 1} = \mu t^2$. For example a type I 3-design [1, 2] for A_1 and A_2, a projective plane of order 3, yields a multiplicative design with $\lambda_1 = 1$, $\lambda_{26} = 16$ and replications 9, 5 and 17.

References

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